SIGNAL RECOVERY USING GOWERS' NORMS

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ABSTRACT. Signal recovery from incomplete or partial frequency information is a fundamental problem in harmonic analysis and applied mathematics, with wide-ranging applications in communications, imaging, and data science. Historically, the classical uncertainty principles—such as those by Donoho and Stark—have provided essential bounds relating the sparsity of a signal and its Fourier transform, ensuring unique recovery under certain support size constraints.

Recent advances have incorporated additive combinatorial notions, notably additive energy, to refine these uncertainty principles and capture deeper structural properties of signal supports. Building upon this line of work, we present a strengthened additive energy uncertainty principle on the finite group $\mathbb{Z}_N^d$, introducing explicit correction terms that measure how far the supports are from highly structured extremal sets like subgroup cosets.

Our main theorems deliver strictly improved bounds over prior results whenever the product of the support sizes differs from the ambient dimension, offering a more nuanced understanding of the interplay between additive structure and Fourier sparsity. Importantly, we leverage these improvements to establish sharper sufficient conditions for unique and exact recovery of signals from partially observed frequencies, explicitly quantifying the role of additive energy in recoverability.

These results advance the theory of discrete signal recovery by providing stronger, more precise guarantees that bridge harmonic analysis and additive combinatorics, and open new pathways for analyzing sparsity and structure in finite discrete settings.

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Date: Summer 2025.

1. Introduction

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a function on the d-dimensional module over $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$, where $N \geq 2$ is an arbitrary positive integer. Our convention is to define the discrete Fourier transform (DFT) of f by $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ such that for $m \in \mathbb{Z}_N^d$

$$\hat{f}(m) := N^{-d/2} \sum_{x \in \mathbb{Z}_N^d} f(x)\chi(-m \cdot x), \tag{1}$$

where $\chi(t) := e^{\frac{2\pi i t}{N}}$. We can then recover f using the inverse Fourier transform

$$f(x) = N^{-d/2} \sum_{m \in \mathbb{Z}_N^d} \hat{f}(m)\chi(m \cdot x). \tag{2}$$

The problem of signal recovery is an inverse problem, where we aim to reconstruct the original function f from a given set of observations of its transform $\hat{f}$. This paper aims to improve the conditions that are sufficient for the unique and exact recovery. The initial conditions are derived directly from the classical Fourier uncertainty principle. Specifically, Donoho and Stark established the following result, formulated here in arbitrary dimensions.

Theorem 1.1 ([DS89]). *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a finite signal in $\mathbb{Z}_N^d$ with* $\mathrm{supp}(f) = E \subseteq \mathbb{Z}_N^d$ *being the set of non-zero entries. Suppose that the set of unobserved frequencies $\{f(m)\}_{m \in \mathbb{Z}_N^d}$ is the set* $\mathrm{supp}(\hat{f}) = M \subseteq \mathbb{Z}_N^d$. *Then the signal f can be recovered uniquely from the observed frequencies if*

$$|E| \cdot |M| < \frac{N^d}{2}. \tag{3}$$

However, Aldahlef at el. recently demonstrated that incorporating additive combinatorial structure, specifically through additive energy, yields stronger uncertainty principles.

The additive energy of a set $A \subset \mathbb{Z}_N^d$ is defined as:

Definition 1.2 (Additive Energy, [AII+25]). *Let $A \subset \mathbb{Z}_N^d$. The* additive energy *of A, denoted by $\Lambda(A)$, is given by*

$$\Lambda(A) = \#\{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 + x_2 = x_3 + x_4\}.$$

In Theorem 1.8. of [AII+25], the authors showed that additive energy could be incorporated into the uncertainty principle in the following way.

Theorem 1.3 (Theorem 1.8, [AII+25]). *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero function with support* $E = \mathrm{supp}(f)$ *and Fourier support* $\Sigma = \mathrm{supp}(\hat{f})$. *Then*

$$N^d \leq |E|\,\Lambda_2(\Sigma)^{1/3}, \tag{4}$$

and, by symmetry,

$$N^d \leq |\Sigma|\,\Lambda_2(E)^{1/3}.$$

Since the expression is symmetric in E and Σ, Fourier inversion allows us to interchange the roles of the support and the Fourier support. When $\Lambda_2(\Sigma) < |\Sigma|^3$, Theorem 1.3 yields a stronger bound than the standard uncertainty principle. In contrast, if $\Sigma = a + H$ is a coset of a subgroup $H \leq \mathbb{Z}_N^d$, then $\Lambda(\Sigma) = |\Sigma|^3$, and the two principles coincide.

Our main contribution is a strengthened uncertainty principle with explicit correction terms that vanish precisely in the extremal case. We prove:

$$N^d \leq |E|\big(\Lambda_2(\Sigma) - C(E,\Sigma)\big)^{1/3}, \tag{5}$$

$$N^d \leq |\Sigma|\big(\Lambda_2(E) - C(\Sigma,E)\big)^{1/3},$$

where $C(E,\Sigma)$ is a non-negative constant depending on E and Σ, which vanishes exactly when the equality $N^d = |E|\,|\Sigma|$ holds. In this case, it follows that both E and Σ achieve maximal additive energy, that is,

$$\Lambda_2(E) = |E|^3 \quad \text{and} \quad \Lambda_2(\Sigma) = |\Sigma|^3.$$

This maximal additive energy characterizes highly structured sets such as cosets of subgroups in $\mathbb{Z}_N^d$. When $N^d \neq |E|\,|\Sigma|$, the additive energy uncertainty inequalities (5) provide a strict improvement over the additive uncertainty principle (4) introduced in [AII+25].

Our main result is the following uncertainty principle.

Theorem 1.4 (Stronger Additive Uncertainty Principle)**.** *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero function (signal) with support* $\mathrm{supp}(f) = E$ *and Fourier support* $\mathrm{supp}(\hat{f}) = \Sigma$*.*

$$N^d \leq |E| \left(\Lambda_2(\Sigma) - |\Sigma|^2 \left(1 - \frac{N^d}{|E||\Sigma|}\right) - |\Sigma|(|\Sigma|-1) \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}}\right)\right)^{1/3} \tag{6}$$

$$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \frac{N^d}{|E||\Sigma|}\right) - |E|(|E|-1) \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}}\right)\right)^{1/3}.$$

This result fits the general form of the new uncertainty principle stated in (5), with explicit constants given by

$$C(E,\Sigma) = |\Sigma|^2 \left(1 - \frac{N^d}{|E||\Sigma|}\right) + |\Sigma|(|\Sigma|-1) \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}}\right),$$

and

$$C(\Sigma,E) = |E|^2 \left(1 - \frac{N^d}{|E||\Sigma|}\right) + |E|(|E|-1) \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}}\right).$$

This theorem sharpens the additive uncertainty principle by subtracting correction terms $C(E,\Sigma)$ and $C(\Sigma,E)$ that quantify how far the supports E and Σ are from this extremal, highly structured case. When the product $|E||\Sigma|$ is strictly greater than N^d, the new inequalities provide a stricter bound than the previously known principle in (4) from [AII+25].

Building on the strengthened additive energy uncertainty principle, we derive a sufficient condition for the unique recovery of a signal when certain frequencies are unobserved. This result quantifies how the additive energy of the unobserved frequency set influences recoverability.

Theorem 1.5 (Additive Recovery Condition)**.** *Suppose frequencies in $S \subseteq \mathbb{Z}_N^d$ are unobserved, and that for all subsets $T \subseteq \mathbb{Z}_N^d$ with $|T| \leq 2|E|$ the additive energy satisfies*

$$\Lambda_2(T) \leq K|T|^\alpha,$$

where $K \geq 0$ and $2 \leq \alpha \leq 3$. If

$$|E|^3\left(\Lambda_2(S) - |E|^3|S|(|S|-1)\left[1 - \sqrt{\frac{K}{(2|E|)^{3-\alpha}}}\sqrt{\frac{N^d}{2|E||S|}}\right]\right.$$
$$\left. - |E|^3|S|^2\left(1 - \frac{N^d}{2|E||S|}\right)\right) < \frac{N^{3d}}{8},$$

then the function f can be uniquely recovered.

Therefore, we get better recovery conditions using the stronger additive uncertainty principle (Theorem 1.4) than was previously derived from the original additive uncertainty principle established by Aldahlef-Iosevich-Iosevich-Jaimangal-Mayeli-Pack in [AII+25].

2. Background on Signal Recovery

Signal recovery is a fundamental problem in applied mathematics and engineering: given incomplete or partially missing information about a signal, can we reconstruct the original signal exactly? In the discrete Fourier setting, this problem is typically phrased as follows. Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a discrete signal, and let $\hat{f}$ denote its discrete Fourier transform. Suppose that the values of $\hat{f}$ are missing on a subset $S \subset \mathbb{Z}_N^d$. The central question is: under what conditions on the signal f and the set of missing frequencies S can the original signal f be recovered exactly from the partial information?

A general theoretical condition that guarantees unique recovery of f was established by Donoho and Stark [DS89]. Suppose we have three functions $f, r, g : \mathbb{Z}_N^d \to \mathbb{C}$ that all agree on the non-missing frequencies $m \notin S$, and assume that their supports satisfy $|\text{supp}(f)| = |\text{supp}(r)| = |\text{supp}(g)| = |E|$. If the product of the support size of f and the size of the missing frequency set satisfies

$$|E||S| < \frac{N^d}{2},$$

then it follows that $r = g = f$. In other words, under this condition, any two functions that match on the known frequencies and have the same support size must be identical, which establishes a uniqueness guarantee for recovery.

The proof, as given in [DS89], elegantly leverages the classical uncertainty principle. The core idea is to consider the difference function $h = r - g$ of any two candidate signals r and g that agree with f on the known frequencies. By construction, $\widehat{h}$ is supported on S and h is supported on at most $2|E|$ points. Applying the uncertainty principle to this nonzero function h leads to the inequality $N^d \leq |\text{supp}(h)|, |\text{supp}(\widehat{h})| \leq 2|E||S|$, which contradicts the assumption $2|E||S| < N^d$. This contradiction forces $h \equiv 0$, proving uniqueness.

While Theorem 1.1 provides a clean sufficient condition for unique identifiability, it is primarily an existence result. The practical task of actually reconstructing the signal f from its partial Fourier measurements is a central problem in the field of sparse recovery or compressed sensing, which is studied in [CRT05, Don06]. The key insight that enabled this field was the realization that the computationally intractable ℓ^0-minimization (which directly seeks the sparsest solution) can be replaced by its convex relaxation, ℓ^1-minimization, under certain conditions. The key idea is that, given the known frequencies $\hat{f}(m)$ for $m \notin S$, the original signal f can be recovered as the unique minimizer of the ℓ^1 norm among all functions that agree with $\hat{f}$ on the known frequencies. Formally, the recovery problem is written as

$$f = \text{argmin}_g \|g\|_{L^1(\mathbb{Z}_N^d)} \quad \text{subject to } \hat{g}(m) = \hat{f}(m) \text{ for all } m \notin S.$$

The guarantee that ℓ^1 minimization indeed recovers f under the condition $|E||S| < N^d/2$ is a central result also established by Donoho and Stark [DS89].

As shown in [DS89], for any function h with $\text{supp}(\widehat{h}) \subseteq S$, the following inequality holds:

$$\|h\|_{L^1(E)} \leq \frac{|E||S|}{N^d} \|h\|_{L^1(\mathbb{Z}_N^d)}.$$

The recovery proof then proceeds by considering a candidate minimizer g and the difference function $h = g - f$. Since $\widehat{h}$ is supported on S, the above inequality applies. Under the assumption $|E||S| < N^d/2$, one can show that the ℓ^1 norm of h on the complement of the true support E^c must be strictly greater than its norm on E. This leads to a contradiction with the assumption that g has a smaller or equal ℓ^1 norm than f, thereby proving that f is the unique minimizer. For the complete and detailed argument, we refer the reader to the original proof in [DS89]. To illustrate these concepts concretely, consider a simple one-dimensional example:

Example 2.1. *Let $f : \mathbb{Z}_4 \to \mathbb{C}$ be given by*

$$f = (1, 0, 0, 2), \quad \textit{so that } \text{supp}(f) = E = \{0, 3\}.$$

The discrete Fourier transform of f is

$$\hat{f} = (\hat{f}(0), \hat{f}(1), \hat{f}(2), \hat{f}(3)) = (3, 1 - i, -1, 1 + i).$$

Suppose that during transmission, the values at frequencies $S = \{1, 2\}$ are lost. Solving the ℓ^1 minimization problem

$$\min_g \|g\|_{L^1(\mathbb{Z}_4)} \quad \textit{subject to } \hat{g}(0) = 3,\ \hat{g}(3) = 1 + i$$

recovers f exactly.

For instance, consider the candidate $g = (2, 1, 0, 0)$, which satisfies the constraints at the known frequencies. Its ℓ^1 norm is

$$\|g\|_{L^1} = |2| + |1| + |0| + |0| = 3,$$

while the true signal has

$$\|f\|_{L^1} = |1| + |0| + |0| + |2| = 3.$$

Slight perturbations of g that maintain the constraints would increase the ℓ^1 norm above that of f, demonstrating that f indeed minimizes the ℓ^1 norm and is therefore recovered exactly.

These results, combining uniqueness guarantees from uncertainty principles and constructive recovery via ℓ^1 minimization, form the foundation for much of the modern theory of signal recovery. For a more comprehensive discussion, including extensions to higher dimensions and alternative reconstruction methods, see [IM25].

The incorporation of additive combinatorial structure into this framework represents a significant theoretical advance. Sets with high additive energy correspond to highly structured configurations such as arithmetic progressions or cosets of subgroups, while sets with low additive energy exhibit less additive structure. The key observation is that less structured sets—those with lower additive energy—admit stronger recovery guarantees.

For a set $A \subseteq \mathbb{Z}_N^d$, the normalized additive energy $\Lambda_2(A)/|A|^3$ ranges from $|A|^{-1}$ (for generic sets) to 1 (for cosets of subgroups). This normalization quantifies how far a set deviates from maximal additive structure, providing a parameter that our strengthened uncertainty principles exploit to improve recovery conditions.

3. Proofs of Theorem 1.4

Proof of Theorem 1.4. Define $1_{x,y,a} = 1_E(x)1_E(y)1_E(x+a)1_E(y+a)$. By Fourier inversion, we have

$$\begin{aligned}\sum_{m\in\Sigma} |\hat{f}(m)|^4 &= N^{-2d} \sum_{m\in\Sigma} \sum_{x,y,z,w\in E} f(x)\overline{f(y)}f(z)\overline{f(w)}\chi(m\cdot(x-y+z-w)) \\ &\leq N^{-d} \sum_{\substack{x+z=y+w \\ x,y,z,w\in E}} f(x)\overline{f(y)}f(z)\overline{f(w)} \\ &\leq N^{-d} \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)|1_{x,y,a}\end{aligned}$$

Therefore,

$$N^{3d} \sum_{m\in\Sigma} |\hat{f}(m)|^4 \leq N^{2d} \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)|1_{x,y,a}$$

By Cauchy Schwarz and another application of Fourier inversion, we have

$$\begin{aligned}&N^{2d} \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)|1_{x,y,a} \\ &\leq N^{2d} \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(x+a)|^2 1_{x,y,a} \\ &\leq \sum_{m_1,\dots,m_4} |\hat{f}(m_1)\hat{f}(m_2)\hat{f}(m_3)\hat{f}(m_4)| \left| \sum_{x,y,a\in\mathbb{Z}_N^d} \chi(x\cdot(m_1-m_2+m_3-m_4)\chi(a\cdot(m_3-m_4))1_{x,y,a} \right| \\ &= \sum_{\substack{m_1,m_2,m_3,m_4 \\ N_1=m_1-m_2 \\ N_2=m_3-m_4}} |\hat{f}(m_1)\hat{f}(m_2)\hat{f}(m_3)\hat{f}(m_4)| \left| \sum_{x,y,a\in\mathbb{Z}_N^d} \chi(x\cdot N_1)\chi((x+a)\cdot N_2)1_{x,y,a} \right|\end{aligned}$$

$\leq I + II + III,$

where now we split into cases

I. $x \neq y$, $a \neq 0$

II. $x = y$

III. $x \neq y$, $a = 0$

In what follows, we bound I, II, and III above by terms of the form $(\dots)\sum_{m\in\Sigma}|\hat{f}(m)|^4$ so that we may cancel to recover an improved uncertainty principle.

Case I: Notice that the sum

$$\sum_{x,y,a\in\mathbb{Z}_N^d \neq y, a\neq 0} 1_E(x)1_E(y)1_E(x+a)1_E(y+a)$$

counts the number of triples (x, y, a) with $x \neq y$ and $a \neq 0$ such that all four points x, y, $x+a$, and $y+a$ lie in E. This is equivalent to counting quadruples $(x, y, z, w) \in E^4$ where $z = x+a$ and $w = y+a$ for some nonzero a, and $x \neq y$. Since $a = z - x = w - y \neq 0$, the conditions $x \neq y$ and $a \neq 0$ imply that $x \neq z$ and $x \neq w$. Thus, we have the identity:

$$\begin{aligned}
&(x, y, a) : x, y, x+a, y+a \in E, , x \neq y, , a \neq 0 \\
&\quad = (x, y, z, w) \in E^4 : z = x+a, , w = y+a, , a \neq 0, , x \neq y \\
&\quad = (x, y, z, w) \in E^4 : x+y = z+w, , x \neq z, , x \neq w.
\end{aligned}$$

This counts all additive quadruples (x, y, z, w) where the pair (z, w) is distinct from both (x, y) and (y, x). The total additive energy $\Lambda_2(E)$ counts all quadruples satisfying $x+y = z+w$. The only quadruples not counted in the above expression are the degenerate cases where (z, w) is identical to (x, y) or (y, x). The number of these trivial quadruples is $|E|^2$ (for $(z, w) = (x, y)$) plus $|E|^2$ (for $(z, w) = (y, x)$), but the $|E|$ quadruples where $x = y$ in both pairs are counted twice. Therefore, the number of non-trivial quadruples is:

$$\Lambda_2(E) - 2|E|^2 + |E|$$

Using the triangle inequality and the above identity, we see that

$$\begin{aligned}
I &= \sum_{m_1,\dots,m_4} |\hat{f}(m_1)\dots\hat{f}(m_4)| \left| \sum_{\substack{x,y,a\in\mathbb{Z}_N^d \\ x\neq y, a\neq 0}} \chi(x\cdot N_1)\chi((a+x)\cdot N_2) 1_{x,y,a} \right| \\
&= \sum_{m_1,\dots,m_4} |\hat{f}(m_1)\dots\hat{f}(m_4)| \sum_{\substack{x,y,a\in\mathbb{Z}_N^d \\ x\neq y, a\neq 0}} 1_{x,y,a} \\
&= (\Lambda_2(E) - 2|E|^2 + |E|) \left(\sum_m |\hat{f}(m)| \right)^4 \\
&\leq (\Lambda_2(E) - 2|E|^2 + |E|)|\Sigma|^3 \left(\sum_m |\hat{f}(m)|^4 \right) \quad \text{(Hölder's inequality)}
\end{aligned}$$

Case II: Observe that

$$\sum_{N_1\in\mathbb{Z}_N^d}\left|\sum_x \chi(x\cdot N_1)1_E(x)\right|^2 = \sum_{N_1\in\mathbb{Z}_N^d}\sum_{x,\tilde{x}}\chi((x-\tilde{x})\cdot N_1)1_E(x) = |E|N^d$$

First, let us establish that for fixed $a, b \in \mathbb{Z}_N^d$ and $m_i \in \Sigma$, we know

$$\left(\sum_{\substack{m_1-m_2=a\\ m_3-m_4=b}} 1\right) = \left(\sum_{m_2\in\Sigma}\mathbf{1}_\Sigma(m_2+a)\right)\left(\sum_{m_4\in\Sigma}\mathbf{1}_\Sigma(m_4+b)\right) \leq |\Sigma|\cdot|\Sigma| = |\Sigma|^2$$

Let $t = x + a$. Then, since $x = y$,

$$\begin{aligned}
II &= \sum_{m_1,\ldots,m_4}|\hat{f}(m_1)\ldots\hat{f}(m_4)|\left|\sum_{x,a\in\mathbb{Z}_N^d}\chi(x\cdot N_1)\chi((a+x)\cdot N_2)1_E(x)1_E(x+a)\right| \\
&= \sum_{N_1,N_2\in\mathbb{Z}_N^d}\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}|\hat{f}(m_1)\ldots\hat{f}(m_4)|\left|\sum_x\chi(x\cdot N_1)1_E(x)\right|\left|\sum_t\chi(t\cdot N_2)1_E(t)\right| \\
&\leq \left(\sum_{N_1,N_2\in\mathbb{Z}_N^d}\left[\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}|\hat{f}(m_1)\ldots\hat{f}(m_4)|\right]^2\right)^{1/2}\times \\
&\quad\times\left(\sum_{N_1,N_2\in\mathbb{Z}_N^d}\left|\sum_x\chi(x\cdot N_1)1_E(x)\right|^2\left|\sum_t\chi(t\cdot N_2)1_E(t)\right|^2\right)^{1/2} \\
&= \left(\sum_{N_1,N_2\in\mathbb{Z}_N^d}\left[\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}|\hat{f}(m_1)\ldots\hat{f}(m_4)|\right]^2\right)^{1/2}\times\left(\sum_{N_1\in\mathbb{Z}_N^d}\left|\sum_x\chi(x\cdot N_1)1_E(x)\right|^2\right) \\
&= |E|N^d\left(\sum_{N_1,N_2\in\mathbb{Z}_N^d}\left[\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}|\hat{f}(m_1)\ldots\hat{f}(m_4)|\right]^2\right)^{1/2} \\
&\leq |E|N^d\left(\sum_{N_1,N_2}\left(\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}|\hat{f}(m_1)\ldots\hat{f}(m_4)|^2\right)\left(\sum_{\substack{m_1-m_2=N_1\\ m_3-m_4=N_2}}1\right)\right)^{1/2}
\end{aligned}$$

$$\leq |E||\Sigma|N^d \left(\sum_{m_1,\ldots,m_4} |\hat{f}(m_1)\ldots\hat{f}(m_4)|^2 \right)^{1/2}$$

$$= |E||\Sigma|N^d \left(\sum_{m} |\hat{f}(m)|^2 \right)^{2}$$

$$\leq |E||\Sigma|^2 N^d \left(\sum_{m} |\hat{f}(m)|^4 \right) \quad \text{(Holder's inequality)}$$

Case III: We have that

$$III = \sum_{m_1,\ldots,m_4} |\hat{f}(m_1)\ldots\hat{f}(m_4)| \left| \sum_{\substack{x,y\in\mathbb{Z}_N^d \\ x\neq y}} \chi(x\cdot(m_1-m_2+m_3-m_4))1_E(x)1_E(y) \right|$$

$$= (|E|-1) \sum_{m_1,\ldots,m_4} |\hat{f}(m_1)\ldots\hat{f}(m_4)| \left| \sum_{x\in E} \chi(x\cdot(m_1-m_2+m_3-m_4)) \right|$$

Let us look at

$$\sum_{m_1,\ldots,m_4} |\hat{f}(m_1)\ldots\hat{f}(m_4)| \left| \sum_{x\in E} \chi(x\cdot(m_1-m_2+m_3-m_4)) \right|$$

$$= \sum_{M\in\mathbb{Z}_N^d} \sum_{m_1-m_2+m_3-m_4=M} |\hat{f}(m_1)\ldots\hat{f}(m_4)| \left| \sum_{x\in E} \chi(x\cdot M) \right|$$

$$\leq \left(\sum_{M\in\mathbb{Z}_N^d} \left(\sum_{m_1-m_2+m_3-m_4=M} |\hat{f}(m_1)\ldots\hat{f}(m_4)| \right)^2 \right)^{1/2}$$

$$\times \left(\sum_{M\in\mathbb{Z}_N^d} \left| \sum_{x\in E} \chi(x\cdot M) \right|^2 \right)^{1/2}$$

$$\leq \left(\sum_{M\in\mathbb{Z}_N^d} \left(\sum_{m_1-m_2+m_3-m_4=M} 1 \right) \left(\sum_{m_1-m_2+m_3-m_4=M} |\hat{f}(m_1)\ldots\hat{f}(m_4)|^2 \right) \right)^{1/2}$$

$$\times |E|^{1/2} N^{d/2}$$

$$\leq (\max_{M} \{(m_1,m_2,m_3,m_4)\in\Sigma^4 \mid m_1-m_2+m_3-m_4=M\})^{1/2}$$

$$\times \left(\sum_{m_1,m_2,m_3,m_4} |\hat{f}(m_1)\dots\hat{f}(m_4)|^2 \right)^{1/2}$$

$$\times |E|^{1/2} N^{d/2}$$

$$\leq (\max_M \{(m_1,m_2,m_3,m_4) \in \Sigma^4 \mid m_1 - m_2 + m_3 - m_4 = M\})^{1/2}$$

$$\times |E|^{1/2}|\Sigma|N^{d/2} \left(\sum_m |\hat{f}(m)|^4 \right)$$

Estimating the number of quadruples

$$|(m_1,m_2,m_3,m_4) \in \Sigma^4 : m_1 + m_3 = m_2 + m_4 + M|.$$

We can say, that by shifting m_4 by M, it is the same as

$$|(m_1,m_2,m_3,m_4) \in \Sigma \times \Sigma \times \Sigma \times (\Sigma + M) : m_1 + m_3 = m_2 + m_4|$$

Now, using bound 5.2 from [AII$^+$25], this quantity is bounded by

$$\left(\Lambda_2(\Sigma)^3\, \Lambda_2(\Sigma + M)\right)^{1/4} = \Lambda_2(\Sigma) \text{ as shifting by } M \text{ does not change the additive energy of } \Sigma$$

Hence, combining all three parts, we obtain

$$N^{3d} \sum_{m\in\Sigma} |\hat{f}(m)|^4 \leq I + II + III$$

$$\leq \left((\Lambda_2(E) - 2|E|^2 + |E|)|\Sigma|^3 + |E||\Sigma|^2N^d + |E|^{1/2}(|E|-1)|\Sigma|\Lambda_2(\Sigma)N^{d/2}\right)$$

$$\times \sum_m |\hat{f}(m)|^4.$$

By canceling $\sum_{m\in\Sigma} |\hat{f}(m)|^4$ from both sides, rearranging terms, and taking a cube root we recover the improved uncertainty principle

$$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \frac{N^d}{|E||\Sigma|}\right) - |E|(|E|-1)\left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}}\sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}}\right)\right)^{1/3}.$$

□

4. Proof of Theorem 1.5

Proof of Theorem 1.5. Assume, there exists $g : \mathbb{Z}_N^d \to \mathbb{C}$ such that $g \neq f$ and

$$\hat{g}(m) = \hat{f}(m) \text{ for } m \notin S \quad \text{and} \quad |\text{supp}(g)| = |\text{supp}(f)| = |E|.$$

Then, let $f = g + h$, where $h : \mathbb{Z}_N^d \to \mathbb{C}$. Because $h = f - g$, then $|\text{supp}(h)| = |T| = |\text{supp}(f) - \text{supp}(g)| \leq 2|E|$. We also know that because $\hat{g}(m) = \hat{f}(m)$ for $m \notin S$, then

$\mathrm{supp}(\hat{h}) = Q \subseteq S$. Hence by Theorem 1.4, if we exchange $\mathrm{supp}(f)$ for $\mathrm{supp}(\hat{f})$, we get

$$N^d \leq |T| \left(\Lambda_2(Q) - |Q|^2 \left(1 - \frac{N^d}{|Q||T|}\right) - |Q|(|Q|-1) \left(1 - \sqrt{\frac{N^d}{|Q||T|}} \sqrt{\frac{\Lambda_2(T)}{|T|^3}}\right)\right)^{1/3}$$

We want to get a contradiction with a condition of the theorem to prove that our f is unique. Let's take both sides of the equation to the third power:

$$N^{3d} \leq |T|^3 \left(\Lambda_2(Q) - |Q|^2 \left(1 - \frac{N^d}{|Q||T|}\right) - |Q|(|Q|-1) \left(1 - \sqrt{\frac{N^d}{|Q||T|}} \sqrt{\frac{\Lambda_2(T)}{|T|^3}}\right)\right)$$

We know that

$$|T|^3(\Lambda_2(Q) - 2|Q^2| + |Q|) \leq 8|E|^2(\Lambda_2(S) - 2|S^2| + |S|),$$

because the quantity in the parenthesis on the left represents the number of non-trivial parallelograms in Q. Since $Q \subseteq S$, then for S that quantity would be larger. For

$$\begin{aligned} &|T|^3 \left(\frac{|Q|N^d}{|T|} + |Q|^2 \sqrt{\frac{N^d}{|Q||T|}} \sqrt{\frac{\Lambda_2(T)}{|T|^3}} + |Q| \left(\sqrt{\frac{N^d}{|Q||T|}} \sqrt{\frac{\Lambda_2(T)}{|T|^3}}\right)\right) \\ &\leq |T|^2|Q|N^d + |T||Q|^{3/2}\sqrt{N^d \cdot \Lambda_2(T)} + |T||Q|^{1/2}\sqrt{N^d \cdot \Lambda_2(T)} \\ &\leq 4|E|^2|S|N^d + 2|E||S|^{3/2}\sqrt{N^d \cdot K \cdot (2|E|)^\alpha} + 2|E||S|^{1/2}\sqrt{N^d \cdot K \cdot (2|E|)^\alpha}. \end{aligned}$$

Hence, if we combine the two, we would get

$$8|E|^3 \left(\Lambda_2(S) - |S|(|S|-1) \left[1 - \sqrt{\frac{K}{(2|E|)^{(3-\alpha)}}} \sqrt{\frac{N^d}{2|E||S|}}\right] - |S|^2 \left(1 - \frac{N^d}{2|E||S|}\right)\right).$$

From the statement of the theorem, we know

$$8|E|^3 \left(\Lambda_2(S) - |S|(|S|-1) \left[1 - \sqrt{\frac{K}{(2|E|)^{(3-\alpha)}}} \sqrt{\frac{N^d}{2|E||S|}}\right] - |S|^2 \left(1 - \frac{N^d}{2|E||S|}\right)\right) < N^{3d}.$$

Hence, combining all the inequalities we get

$$N^{3d} < N^{3d},$$

which is a contradiction, which means f is unique. □

5. Conclusion

We have presented strengthened additive energy uncertainty principles that provide strict improvements over previously known bounds whenever the product of support sizes differs from the ambient dimension. Our main theoretical contribution lies in the explicit correction terms $C(E, \Sigma)$ and $C(\Sigma, E)$ that quantify precisely how the additive structure of supports affects the uncertainty relations.

The key innovation in our approach is the decomposition of Fourier-analytic expressions into components corresponding to different types of additive configurations—non-degenerate parallelograms, degenerate line configurations, and point pairs. This decomposition, combined with careful applications of Hölder's inequality and additive combinatorial bounds, yields the improved uncertainty principles.

From a practical perspective, Theorem 1.5 demonstrates that our theoretical improvements translate directly into enhanced recovery guarantees. For signals and missing frequency sets with sub-maximal additive energy, our conditions allow recovery from strictly more missing data than permitted by classical theory or previous additive energy-based bounds.

The correction terms in our uncertainty principles exhibit several notable properties. First, they vanish precisely when $|E||\Sigma| = N^d$ and both sets achieve maximal additive energy, confirming that cosets of subgroups represent the extremal case. Second, they increase as the additive energy decreases, thereby quantifying how less structured sets enable better recovery. Finally, they depend on both the support and the Fourier support, capturing the connection between structure in both domains.

6. Future work

Several directions for further research arise naturally from our results. Inspired by the identity $\Lambda_2(E) = |E|^2$, it is natural to investigate uncertainty principles involving higher-order quantities such as

$$\Lambda_{k+1}(E) - \Lambda_k(E)(\dots),$$

which enumerate non-degenerate $(k+1)$-dimensional parallelepipeds. Understanding how these refinements interact with Fourier concentration may yield sharper structural bounds.

Beyond the classical additive energy, one may consider alternative frameworks such as the number of solutions to

$$a + b + c + d = e + f + g + h,$$

which arise from the Fourier expansion of

$$\sum_{x \in \mathbb{Z}_N^d} |\hat{f}(x)|^8.$$

These higher-order energies could lead to new structural parameters for uncertainty principles.

On the algorithmic side, while ℓ_1- and ℓ_2-minimization methods are standard in compressed sensing, it is natural to ask whether analogous recovery guarantees can be obtained via minimization in the U_k-norm. Developing such algorithms would directly leverage higher-order additive structure.

More broadly, extensions to other finite abelian groups and eventually to continuous domains remain open problems. The adaptation to noisy measurements, rather than complete erasures, is also essential for applications in signal processing.

In summary, the incorporation of higher-order additive energies, alternative combinatorial invariants, and U_k-based algorithms may lead to substantial refinements of uncertainty principles and recovery guarantees.

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