Signal Recovery using Gowers' Norms

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Advised by Prof. Alex Iosevich and Prof. Eyvindur Palsson

SMALL REU 2025; YMC

Introduction

The Signal is Broken. Can You Fix It?

"Hi"

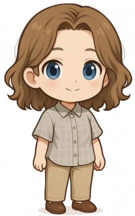
↓

0100100001101001

↓

Received:

0100100001101001



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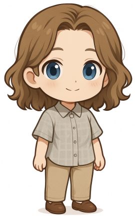
$\downarrow$

0 1 0 0 1 0 0 0 0 1 1 0 1 0 0 1

$\downarrow$



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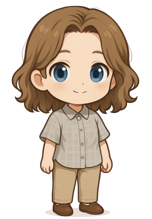

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No way to restore the original signal :(

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DFT:

$$[\ \widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5, \widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}, \widehat{f}_{13}, \widehat{f}_{14}, \widehat{f}_{15}\ ]$$

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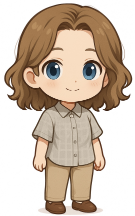

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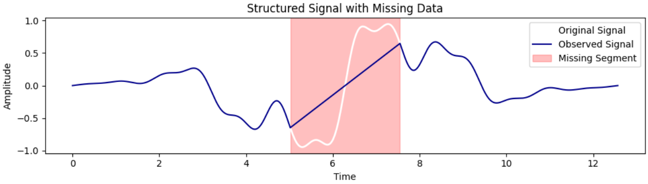
$\downarrow$

Received:

$$[\ \blacksquare, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \blacksquare, \blacksquare, \widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}, \blacksquare, \widehat{f}_{14}, \widehat{f}_{15}\ ]$$

The Signal is Broken. Can You Fix It?

- You receive only part of a signal/frequencies - the rest is missing.



(Illustrative example of a corrupted signal)

- Is it possible to reconstruct the full message?

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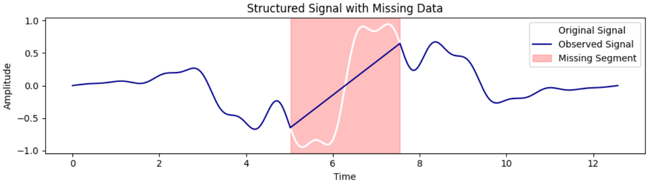

(Illustrative example of a corrupted signal)

- Is it possible to reconstruct the full message?
- Sufficient conditions for reconstruction? What if you know the signal/frequency is "structured"?

Fourier Analysis and Additive Combinatorics

We'll use tools from **Fourier Analysis** and **Additive Combinatorics** to find out.

Fourier Transform

Definition (Discrete Fourier Transform)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the **normalized DFT** is:

$$\widehat{f}(k) := \frac{1}{\sqrt{N^d}} \sum_{n \in \mathbb{Z}_N^d} f(n)\chi(-kn),$$

where $\chi(x) = e^{-2\pi i k \cdot x/N}$. Then, the inverse transform formula follows:

$$f(n) = \frac{1}{\sqrt{N^d}} \sum_{k \in \mathbb{Z}_N^d} \widehat{f}(k)\chi(kn).$$

Fourier Transform Notation

- We will call an arbitrary function $f : \mathbb{Z}_N^d \to \mathbb{C}$ a **signal**.
- We will call an arbitrary function's fourier transform $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ a **frequency**.

Support

Definition (Support)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a function.

- The **support** of f, denoted $\mathrm{supp}(f)$, is the set of points where f is nonzero:

$$\mathrm{supp}(f) = \{x \in \mathbb{Z}_N^d : f(x) \neq 0\}.$$

- The support of the **discrete Fourier transform** $\widehat{f}$ is similarly defined as:

$$\mathrm{supp}(\widehat{f}) = \{\xi \in \mathbb{Z}_N^d : \widehat{f}(\xi) \neq 0\}.$$

Classical Uncertainty Principle

- Many of you are familiar with Heisenberg uncertainty principle:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

It turns out, there is a discrete version in Fourier Analysis!

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Theorem (Classical Uncertainty Principle)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero function with support $\mathrm{supp}(f) \subseteq \mathbb{Z}_N^d$. Let $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ denote the discrete Fourier transform of f, with support $\mathrm{supp}(\widehat{f}) \subseteq \mathbb{Z}_N^d$. Then the following inequality holds:

$$|\mathrm{supp}(f)| \cdot |\mathrm{supp}(\widehat{f})| \geq N^d.$$

Discrete L_p-norm

Definition (L_p Norm)

For a function $f : \mathbb{Z}_N \to \mathbb{C}$, the L_p norm is defined as:

$$\|f\|_{L_p(\mathbb{Z}_N^d)} := \begin{cases} \left(\frac{1}{N^d}\sum_{n=0}^{N^d-1} |f(n)|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{0\leq n<N^d} |f(n)| & \text{if } p = \infty. \end{cases}$$

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Theorem (Holder's Inequality)

For a function $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \cdot \|g\|_{L_q}.$$

Unique Recovery Principle

Theorem (Classical Recovery Condition, Donoho et al., 1989, [DS89])

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ supported in $E \subset \mathbb{Z}_N^d$. Suppose that $\widehat{f}$ is transmitted but the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, where $S \subset \mathbb{Z}_N^d$, with

$$|E| \cdot |S| < \frac{N^d}{2}. \tag{1}$$

Then f can be recovered exactly and uniquely. Moreover,

$$f = \arg\min_g \|g\|_{L_1(\mathbb{Z}_N^d)} \tag{2}$$

with the constraint $\widehat{f}(m) = \widehat{g}(m), \ m \notin S$.

Recap: What Have We Learned So Far?

- We model signals as functions on $\mathbb{Z}_N^d$, and study their behavior using the **Discrete Fourier Transform**.
- The **Fourier transform** reveals the frequency structure of a signal.
- The **Uncertainty Principle** tells us that a function and its Fourier transform cannot both be highly localized:

$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\widehat{f})| \geq N^d.$$

- We use L_p **norms** and **Hölder's inequality** to measure signal magnitude and relate functions.
- Under certain conditions on the size of the support, we can **exactly recover** a signal from incomplete frequency data by minimizing its L_1 norm.

Gower's Norms

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Definition

The **additive energy** of a set $A \subset \mathbb{Z}^d$ is defined as:

$$\Lambda_2(A) := \left|\left\{(x_1, x_2, x_3, x_4) \in A^4 : x_1 + x_2 = x_3 + x_4\right\}\right|,$$

where $|\cdot|$ denotes the cardinality of the set.

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Definition (Gowers U_2-norm)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the Gowers U^2 norm is defined as:

$$\|f\|_{U_2}^4 := \mathbb{E}_{x,h_1,h_2}\left[f(x)\overline{f(x+h_1)f(x+h_2)}f(x+h_1+h_2)\right].$$

What Does the Gowers U_2 Norm Measure?

Intuition

The U_2 norm detects **patterns** in a signal. It is high if the signal contains many parallelogram-like structures:

$$f(x),\ f(x+h_1),\ f(x+h_2),\ f(x+h_1+h_2).$$

- Random signals $\to$ low U_2
- Structured signals (e.g., arithmetic patterns) $\to$ high U_2

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- Random signals $\rightarrow$ low U_2
- Structured signals (e.g., arithmetic patterns) $\rightarrow$ high U_2
- Gowers norms help quantify how **non-random** a signal is.

Additive Uncertainty Principle

Theorem (Additive Uncertainty Principle - Iosevich-Mayeli '25 [AII$^+$25])

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero signal with support in E, and let $\widehat{f}$ denote its Fourier transform with support in Σ. Then for any $\alpha \in [0, 1]$,

(i) $N^d \leq (|E| \cdot \Lambda_2^{\frac{1}{3}}(\Sigma))^{1-\alpha} \cdot (\Lambda_2^{\frac{1}{3}}(E) \cdot |\Sigma|)^{\alpha}$.

To prove part (i), it is sufficient to establish the inequality

$$N^d \leq |\Sigma| \cdot \Lambda_2^{\frac{1}{3}}(E).$$

The inequality $N^d \leq |E| \cdot \Lambda^{\frac{1}{3}}(\Sigma)$ follows by reversing the roles of E and Σ, and the general case follows from these two by writing $N^d = N^{d(1-\alpha)} \cdot N^{d\alpha}$, $0 \leq \alpha \leq 1$. [AII$^+$25]

Additive Uncertainty Principle: Improved

We were able to improve the additive uncertainty principle:

Theorem (SMALL 2025)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Suppose f is supported on $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported on $\Sigma \subset \mathbb{Z}_N^d$. Then we have the uncertainty principle:

(i) $$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}} \right) \right)^{1/3}$$

(ii) $$N^d \leq |\Sigma| \left(\frac{\sqrt{B_\Sigma |E| (\Lambda_2(E) - |E|^2)}}{|\Sigma|} + |E|^2 \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right)^{1/3},$$

where $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)|$.

Ask us what's the difference!

Comparing Two Additive Energy Inequalities

Classical Additive Inequality

$$N^d \leq |\Sigma| \cdot \Lambda_2^{\frac{1}{3}}(E)$$

- Simple, elegant
- Balances **sumset size** and **additive energy**

Refined Inequality

$$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}} \right) \right)^{1/3}$$

- Adds **correction term** for extremal structure
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Note: Additive energy satisfies $|E|^2 \leq \Lambda_2(E) \leq |E|^3$

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Note: Additive energy satisfies $|E|^2 \leq \Lambda_2(E) \leq |E|^3$

Takeaway: Both inequalities show how *structure constrains sumsets*, with the refined version offering sharper bounds in structured regimes.

Unique Recovery Principle

Theorem (Additive Recovery Condition)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ supported in $E \subset \mathbb{Z}_N^d$. Suppose that $\widehat{f}$ is transmitted but the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, where $S \subset \mathbb{Z}_N$, with

$$|E| \cdot \Lambda_2^{1/3}(S) < \frac{N^d}{2}. \qquad (3)$$

Then f can be recovered exactly and uniquely.

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Then f can be recovered exactly and uniquely.

Note: When a set of missing frequencies has a low additive energy, we expect $\Lambda_2(S) \sim |S|^2$. So, signal can be recovered uniquely if

$$|E||S|^{2/3} \lesssim N^d \tag{4}$$

Improved Unique Recovery Principle

Theorem (New Recovery Condition)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ supported in $E \subset \mathbb{Z}_N^d$. Suppose that $\widehat{f}$ is transmitted but the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, where $S \subset \mathbb{Z}_N$. For convinience, let's say

$$\Lambda_2(T) \leq |T|^\alpha$$
$$2 \leq \alpha \leq 3, \forall T \subset \mathbb{Z}_N^d : |T| \leq 2|E|.$$

If

$$|E|^3 \Lambda_2(S) - |E|^3 |S|^2 \left(1 - \frac{1}{(2|E|)^{(3-\alpha)/2}} \sqrt{\frac{N^d}{2|E||S|}}\right) < \frac{N^{3d}}{8}. \qquad (5)$$

Then f can be recovered exactly and uniquely.

Comparison of recovery conditions

So far we have three recovery conditions:

Recovery Conditions summary

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3. (SMALL)

$$|E|^3\Lambda_2(S) - |E|^3|S|^2\left(1 - \frac{1}{(2|E|)^{(3-\alpha)/2}}\sqrt{\frac{N^d}{2|E||S|}}\right) < \frac{N^{3d}}{8}$$

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$$|E|^3\Lambda_2(S) - |E|^3|S|^2\left(1 - \frac{1}{(2|E|)^{(3-\alpha)/2}}\sqrt{\frac{N^d}{2|E||S|}}\right) < \frac{N^{3d}}{8}$$

A new result **(3)** is stronger than **(2)** when $|E||S| \geq N^d/2$.

Refined Additive Uncertainty Principle

Theorem (SMALL 2025)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Suppose f is supported on $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported on $\Sigma \subset \mathbb{Z}_N^d$. Then we have the uncertainty principle:

(i) $$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(\Sigma)}{|\Sigma|^3}} \right) \right)^{1/3}$$

(ii) $$N^d \leq |\Sigma| \left(\frac{\sqrt{B_\Sigma |E| (\Lambda_2(E) - |E|^2)}}{|\Sigma|} + |E|^2 \sqrt{\frac{N^d}{|E|||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right)^{1/3},$$

where $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)|$.

Refined Additive Uncertainty Principle: Proof

Define

$$1_{x,y,a} := 1_E(x)\,1_E(y)\,1_E(x+a)\,1_E(y+a).$$

We begin by applying the Cauchy-Schwarz inequality to the following sum:

$$\sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)| \cdot 1_{x,y,a}$$

$$\leq \left(\sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(x+a)|^2 \cdot 1_{x,y,a}\right)^{1/2} \left(\sum_{x,y,a\in\mathbb{Z}_N^d} |f(y)f(y+a)|^2 \cdot 1_{x,y,a}\right)^{1/2}$$

$$= \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(x+a)|^2 \cdot 1_{x,y,a}$$

$$= (*)$$

Refined Additive Uncertainty Principle: Proof

$$
\begin{aligned}
(*) = N^{-2d} \sum_{m_1,\ldots,m_4} & \widehat{f}(m_1)\overline{\widehat{f}(m_2)}\widehat{f}(m_3)\overline{\widehat{f}(m_4)} \\
& \times \sum_{x,y,a} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4))\cdot 1_{x,y,a} \\
\leq N^{-2d} \sum_{m_1,\ldots,m_4} & |\widehat{f}(m_1)\widehat{f}(m_2)\widehat{f}(m_3)\widehat{f}(m_4)| \\
& \times \left| \sum_{\substack{x,y,a \\ a=0}} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4))1_{x,y,a} \right| \\
+ N^{-2d} \sum_{m_1,\ldots,m_4} & |\widehat{f}(m_1)\widehat{f}(m_2)\widehat{f}(m_3)\widehat{f}(m_4)| \\
& \times \left| \sum_{\substack{x,y,a \\ a\neq 0}} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4))1_{x,y,a} \right| \\
=: S_1 + S_2 &
\end{aligned}
$$

Refined Additive Uncertainty Principle: Proof

By applying Cauchy-Schwarz and Hölder's inequalities as well as exploiting the properties of $\chi(x)$, we get the following inequalities

$$S_1 \leq N^{-2d}|E|^2|\Sigma|^3\sqrt{\frac{N^d}{|E||\Sigma|}}\sqrt{\frac{\Lambda_2(E)}{|E|^3}}\left(\sum_m |\widehat{f}(m)|^4\right)$$

$$S_2 \leq N^{-2d}(\Lambda_2(E)-|E|^2)|\Sigma|^3\left(\sum_{m\in\Sigma}|\widehat{f}(m)|^4\right) \qquad \text{or}$$

$$\leq N^{-2d}\frac{\sqrt{B_\Sigma|E|(\Lambda_2(E)-|E|^2)}}{|\Sigma|}|\Sigma|^3\left(\sum_{m\in\Sigma}|\widehat{f}(m)|^4\right),$$

where $B_\Sigma = |\Sigma-\Sigma||(\Sigma+\Sigma)-(\Sigma+\Sigma)|$.

Refined Additive Uncertainty Principle: Proof

We are getting the statement of our new theorem by estimating the original $N^{3d} \cdot ||f||_{U_2}^4$ from below:

$$\sum_{x,y,a \in \mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)| \cdot 1_{x,y,a} \geq N^d \sum_m |\widehat{f}(m)|^4$$

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$$N^d \sum_m |\widehat{f}(m)|^4 \leq N^{-2d} |\Sigma|^3 \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right) \right) \sum_m |\widehat{f}(m)|^4$$

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$$\boxed{N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right) \right)^{\frac{1}{3}}}$$

Closing

Higher Gower Norms

There is a generalisation of Gower U_k norms for k greater than 2:

Definition (Gowers U_k-norm)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the Gowers U_k norm is defined as:

$$\|f\|_{U_k}^{2^k} N^{(k-1)d} := \sum_{x, h_1, \ldots h_k \in \mathbb{Z}_N^d} \prod_{w_j \in \{0,1\}} J^{w_1 + \ldots w_k} f(x + w_1 h_1 + \ldots w_k h_k).$$

Where J denotes complex conjugation.

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Note

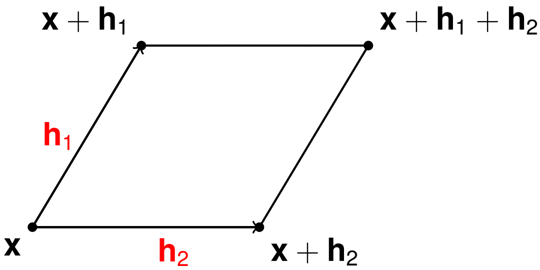
If we set f to be the indicator function of a set E, then

$$\Lambda_2(E) = N^d \|\mathbf{1}_E\|_{U_2}^4 \tag{6}$$

Inspired by this identity, we can define k-additive energy of a set E as:

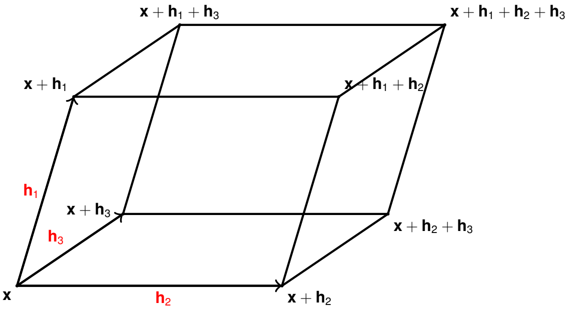
$$\Lambda_k(E) = N^{(k-1)d} \|\mathbf{1}_E\|_{U_k}^{2^k} \tag{7}$$

For U_2 norm, we are counting number of parallelograms:



Intuition for U_k additive energy

For U_3 norm, we are counting number of 3-D parallelepipeds:



We expect higher Λ_k energies to capture **more information** about the additive structure of a support!

Future Work

- Inspired by a fact that $\Lambda_1(E) = |E|^2$ we seek to find uncertainty principle that invokes a term

$$\Lambda_{k+1}(E) - \Lambda_k(E)(\ldots)$$

To account number of non degenerate $k+1$ dimensional parallelepipeds.

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- Consider other additive energy frames, such as number of tuples $a+b+c+d = e+f+g+h$, which naturally arise from Fourier Transform of
$$\sum_{x \in \mathbb{Z}_N^d} |\hat{f}(x)|^8.$$

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- It is known that there are L_1 and L_2 minimisation algorithms for signal recovery. Is it possible to find U_k norm minimisation algorithm?

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SMALL 2025: Additive Energy Uncertainty Principle

Assume, $|\Sigma| \leq N^{d/3}$ and $|\Sigma| \leq (|E| - 1)^{1/4}$, then if we compare

$$\frac{\sqrt{B_\Sigma |E|(\Lambda_2(E) - |E|^2)}}{|\Sigma|} \quad \text{and} \quad \Lambda_2(E) - |E|^2$$

It is the same as comparing

$$B_\Sigma |E| \quad \text{and } |\Sigma|^2(\Lambda_2(E) - |E|^2)$$

We know that $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)| \leq |\Sigma|^6$. We also know that $\Lambda_2(E) - |E|^2 \geq |E|^2 - |E|$, so

$$B_\Sigma |E| \leq |\Sigma|^6 |E| \leq |\Sigma|^2 |E||\Sigma|^4 \leq |\Sigma|^2(\Lambda_2(E) - |E|^2)$$

Hence, the second inequality is stronger than the first one.